# Riemann matrices for the Hyperbolic Curves (The research of geometric structures in quantum information based on Operator Theory and related topics)



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数理解析研究所講究録
第2033巻 2017年 70-82


# Riemann matrices for the Hyperbolic Curves

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**Abstract.**

The Riemann matrix of the Riemann surface determined as the non-singular model of the plane algebraic curve $F_T(x,y,z) = \det(x\Re(T)+y\Im(T)+zI) = 0$ of a complex matrix $T$ provides a new invariant for us to study the numerical ranges of matrices.

## 1.Numerical range of a matrix

Let $T$ be an $n \times n$ complex matrix. The numerical range of $T$ is defined as

$$W(T) = \{\langle T\xi, \xi\rangle : \xi \in \mathbf{C}^n, \xi^*\xi = 1\},$$

where $\mathbf{C}^n$ is considered as the set of column vectors. This set is a compact convex set by the Toeplitz-Hausdorff theorem (1918-1919). The spectrum $\sigma(T)$ of $T$ is contained in this set $W(T)$. Its support line

$$\max\{\Re(e^{-i\theta}z) : z \in W(T)\} = h(\theta)$$

is characterized as

$$h(\theta) = \max\sigma(\Re(e^{-i\theta}T))$$

for any angle $0 \leq \theta \leq 2\pi$. It is known that $W(T)$ contains an interior point in the Gaussian plane $\mathbf{C}$ unless $T$ is normal and the spectrum $\sigma(T)$ lies on a line. In such an exceptional case $T$ is called essentially Hermitian. The numerical rage $W(T)$ satisfies

$$W(T + \lambda I) = \lambda + W(T)$$

for any complex number $\lambda$. We assume that $T$ is not essentially Hermitian and hence the range $W(T)$ contains an interior point. By using the above property, we assume that 0 is an interior point of $W(T)$. Under this assumption, we set

$$K(T) = \{(x, y) \in \mathbf{R}^2 : x\Re(z) + y\Im(z) + 1 \geq 0 \,\text{for any}\, z \in W(T)\}.$$

Then this set $K(T)$ is a compact convex set in the plane $\mathbf{R}^2$ and the set $W(T)$ is characterized as

$$W(T) = \{X + iY : (X, Y) \in \mathbf{R}^2, Xx + Yy + 1 \geq 0 \,\text{for any}\, (x, y) \in K(T)\}.$$

In the case $T$ is an essentially Hermitian matrix, the treatment of $W(T)$ is so easy. In the case $T$ is not essentially Hermitian, if we restrict our attention to the boundary of $W(T)$, we can determine its boundary by using the boundary of the range $K(T)$. For the determination of the boundary of $W(T)$, we do not have to assume that 0 is an interior point of $W(T)$. We define the ternary form $F_T(x, y, z)$ associated to $T$ by

$$F_T(x, y, z) = \det(x\Re(T) + y\Im(T) + zI_n).$$

where $\Re(T) = (T + T^*)/2$, $\Im(T) = (T - T^*)/(2i)$. If $T$ is an essentially Hermitian matrix, the set

$$\{(x, y) \in \mathbf{R}^2 : F_T(x, y, 1) = 0\}$$

composed of finite number of parallel straight lines. If $T$ is non-Hermitian normal matrix satisfying the condition that 0 is an interior point of $W(T)$, then the set $K(T)$ is a compact convex set surrounded by a convex polygon. Each edge of $K(T)$ on the line $a_j x + b_j y + 1 = 0$ corresponds to an eigenvalue $a_j + \sqrt{-1}b_j$ of $T$. In 1951, A German mathematician Kippenhahn introduced an algebraic curve method to treat the boundary of the range $W(T)$ by using the ternary form $F_T(x, y, z)$ or its associative curve

$$C(T) = \{[(x, y, z)] \in \mathbf{CP}^2 : F_T(x, y, z) = 0\}$$

in the complex projective space $\mathbf{CP}^2$. This space is the quotient space

$$\{(x, y, z) \in \mathbf{C}^3 : (x, y, z) \neq (0, 0, 0)\}$$

with respect to the equivalence relation $(x, y, z) \equiv (x', y', z')$ defined by $x' = kx, y' = ky, z' = kz$ for some non-zero scalar $k \in \mathbf{C}$. The factor decomposition of the ternary form $F_T(x, y, z)$ is unique in the polynomial ring $\mathbf{C}[x,y,z]$ up to constant factors. If $F_T(x,y,z)$ is expressed as the product of two (necessarily homogeneous) polynomials $F_1(x,y,z)$ and $F_2(x,y,z)$, then the curves $F_1(x,y,z)=0$ and $F_2(x,y,z)=0$ have some common points $P_j=(x_j,y_j,z_j)\neq(0,0,0)$ at which first derivatives satisfy

$$F_x(x_j,y_j,z_j)=F_y(x_j,y_j,z_j)=F_z(x_j,y_j,z_j)=0$$

with the equation $F(x_j,y_j,z_j)=0$. Such points $(x_j,y_j,z_j)$ are called *singular points* of the curve $C(T):F_T(x,y,z)=0$. Even if $F_T(x,y,z)$ is irreducible in the polynomial ring $\mathbf{C}[x,y,z]$, the curve $C(T)$ may have singular points. We provides two typical examples. Let

$$N_1=\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix},N_2=\begin{pmatrix}0&1&0&1\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}.$$

Then the polynomials $4F_{N_1}(x,y,z)=x^3+xy^2-3(x^2+y^2)z+4z^3$ and $4F_{N_2}(x,y,z)=x^2y^2+y^4-4(x^2+y^2)z^2+4z^4$ are irreducible in the polynomial ring. But the curve $C(N_1)$ has a singular point at $(x,y,z)=(2,0,1)$. The curve $C(N_2)$ has a pair of singular points at $(x,y,z)=(0,\pm\sqrt{2},1)$. If the form $F_T(x,y,z)$ has a repeated factor $H(x,y,z)$, every point of the curve $H(x,y,z)=0$ is a singular point of the curve $F_T(x,y,z)=0$, and hence the curve $C(T)$ has infinite many singular points. We assume that $F_T(x,y,z)$ is multiplicity free. Under this assumption, at almost every point $(x_1,y_1,z_1)\neq(0,0,0)$ of the curve $C(T)$ [except for finite many singular point], the curve $C(T)$ has the unique tangent

$$F_x(x_1,y_1,z_1)x+F_y(x_1,y_1,z_1)y+F_z(x_1,y_1,z_1)z=0.$$

We consider the closure of the set

$$\{[(F_x(x_1,y_1,z_1),F_y(x_1,y_1,z_1),F_z(x_1,y_1,z_1))]\in\mathbf{CP}^2:$$

$$(x_1,y_1,z_1)\,\text{is a non}-\text{singular point of}\,C(T)\}.$$

This set is also an algebraic curve. The closure of this set is expressed as

$$\{[(X,Y,Z)]\in\mathbf{CP}^2:G_T(X,Y,Z)=0\}$$

for some real ternary form $G_T(X,Y,Z)$. If $F_T$ is an irreducible polynomial of degree $n$ and the curve $C(T)$ has no singular points, then the degree of the polynomial of $G_T$ is $n(n-1)$. If $C(T)$ is an order $n$ curve with no singular point, then the boundary of $W(T)$ is given by

$$\partial W(T) = \{X + iY : (X, Y) \in \mathbf{R}^2, G_T(X, Y, 1) = 0\}.$$

If $F_T$ is a general multiplicity free curve, then Kippenhahn's theorem is expressed as

$$W(T) = \mathrm{Conv}(\{X + iY : (X, Y) \in \mathbf{R}^2, G_T(X, Y, 1) = 0\}).$$

In fact, boundary points of $W(T)$ are classified into the two classes. The first class consists of points $X+iY$ for which $(X, Y) \in \mathbf{R}^2$ satisfies $G_T(X, Y, 1) = 0$. The second class consists of line segments $[X_1 + iY_1, X_2 + iY_2]$ for which these segments are extended to the common tangent line of the curve $G_T(X, Y, 1) = 0$ at $(X_1, Y_1, 1), (X_2, Y_2, 1)$.

The real part of the above curve

$$\{[(x, y, z)] \in \mathbf{RP}^2 : F_T(x, y, z) = 0\},$$

or the real affine part of the curve

$$\{(x, y) \in \mathbf{R}^2 : F_T(x, y, 1) = 0\}$$

also attract our attentions.

Kippenhahn provided a birational method to treat $W(T)$ by using $F_T$. We shall consider the compact *Riemann surface* defined as a non-singular model of the curve $F_T(x, y, z) = 0$.

## 2. Compact Riemann surfaces

A compact Riemann surface $S$ is an (orientable) complex 1-dimensional analytic manifold. The complete topological invariant of compact Riemann surfaces is given by its *genus* $g$. The genus of $S$ is the number of holes of $S$ realized as a topological space in $\mathbf{R}^3$.

Example 1: Riemann sphere $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$: $g = 0$.

Example 2: Torus; $\mathbf{C}/\mathbf{Z}^2$: $g = 1$.

Example 3: Doble torus: $g = 2$.

Example 4: Triple torus: $g = 3$.

# 3. Homology group, Riemann matrix

We consider a metric invariant of a Riemann surface $S$. Let $T$ be an $n \times n$ complex matrix for which the form $F_T$ is irreducible. By blowing up of singular points of the curves $F_T(x, y, z) = 0$, we obtain a compact Riemann surface with genus $g \leq (n-1)(n-2)/2$. For instance, a generic $4 \times 4$ matrix $T$ has an associated curve $F_T(x, y, z) = 0$ which is a Riemann surface with $g = 3$.

We shall consider a general compact Riemann surface $S$ with genus $g$. If $g$ is 0, the fundamental group $\pi(S)$ of $S$ is the trivial group, that is, $S$ is simply connected. If $g$ is 1, the space $S$ is homeomorphic to the torus $\mathbf{R}^2/\mathbf{Z}^2$ and hence the fundamental group $\pi(S)$ of $S$ is isomorphic to the abelian group $\mathbf{Z}^2$. We shall treat the case $g \geq 2$. Then the group $\pi_1(S)$ is isomorphic to the free group $F_g$ with $g$ generators. We shall consider the integration of holomorphic differential 1-forms $\omega$ on $S$ over closed oriented path $\gamma$ on $S$:

$$\oint_\gamma \omega. \tag{3.1}$$

The set $H^1(S)$ of all holomorphic differential 1-forms on $S$ is a complex $g$-dimensional vector space if the genus of $S$ is $g$. The space $H^1(S)$ viewed as an abelian group is called the *cohomology group* of $S$. We shall provide a concrete example.

**Example.** Let

$$T = \begin{pmatrix} 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the form $F_T(x, y, z)$ is given by

$$F_T(x, y, z) = 16y^4 + (20x^2 - 12z^2)y^2 + 4x^4 - 12x^2z^2 + z^4.$$

By using the first derivative $F_y(x, y, z) = 8y(5x^2 + 8y^2 - 3)$, we find that a basis of the vector space $H^1(S)$ of this non-singular curve $F_T(x, y, 1) = 0$ is given by

$$\omega_1 = \frac{1}{8}\frac{dx}{y(5x^2 + 8y^2 - 3)}, \quad \omega_2 = \frac{1}{8}\frac{dx}{5x^2 + 8y^2 - 3}, \quad \omega_3 = \frac{1}{8}\frac{xdx}{y(5x^2 + 8y^2 - 3)}$$

([5]). By Cauchy's theorem, if a closed path $\gamma$ is deformed to another closed path $\gamma'$ by a 1-parameter family of continuous maps, that is, $\gamma$ is homotopic to $\gamma'$, the equation

$$\oint_{\gamma} \omega = \oint_{\gamma'} \omega, \tag{3.2}$$

holds for any holomorphic form $\omega \in H^1(S)$. We consider an oriented closed path $\gamma$ on $S$ as a continuous (rectifiable ) curve of $[0,1]$ into $S$ satisfying the condition $\gamma(0) = \gamma(1) = P_0$ for a given base point $P_0$ of $S$. For two closed paths $\gamma, \gamma'$, we define its product $\gamma' \circ \gamma$ by

$$\gamma' \circ \gamma(t) = \gamma(2t), \gamma' \circ \gamma(t + (1/2)) = \gamma'(2t)$$

for $0 \leq t \leq 1$. Then the equation

$$\oint_{\gamma' \circ \gamma} \omega = \oint_{\gamma \circ \gamma'} \omega = \oint_{\gamma} \omega + \int_{\gamma'} \omega, \tag{3.3}$$

holds for any $\omega \in H^1(S)$ and closed oriented paths $\gamma, \gamma'$ on $S$. By the equations (3.2), (3.3), closed oriented paths on $S$ form an abelian group

$$H_1(S, \mathbf{Z}) = \pi_1(S)/N$$

where $N$ is the commutator subgroup of $\pi_1(S)$. This group $H_1(S)$ is called the homology group of $S$. The homology group $H_1(S)$ is isomorphic to $\mathbf{Z}^{2g}$ if the genus of $S$ is $g$. We define the intersection index $\gamma \cdot \gamma'$ of two closed paths $\gamma$ and $\gamma'$. We assume that these two paths have common points at $P_1, P_2, \ldots, P_m$. The intersection index $\gamma \cdot \gamma'$ is the sum of the local intersection indices

$$(\gamma \cdot \gamma')_{P_j}$$

for $j = 1, 2, \ldots, m$. If two paths $\gamma$, $\gamma'$ have common tangent at $P_j$, then $(\gamma \cdot \gamma')_{P_j} = 0$. We consider the respective tangent vectors $v_1, v_2$ of the paths $\gamma, \gamma'$ at $P_j$. If the outer product $v_1 \times v_2$ of $v_1, v_2$ points out the surface $S$, then $(\gamma \cdot \gamma')_{P_j} = +1$. If $v_1 \times v_2$ points into the surface $S$, then $(\gamma \cdot \gamma')_{P_j} = -1$. It is known that the homology group $H_1(S : \mathbf{Z})$ of a Riemann surface with genus $g \geq 1$ has a canonical basis

$$\{a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g\}$$

which satisfy the relation

$$a_i \cdot a_j = b_i \cdot b_j = 0, \quad a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij}$$

( $i, j = 1, 2, \ldots, g$), where $\delta_{ij}$ is the Kronecker delta. Tretkoff [13] provided an algorithm to construct a canonical basis of the homology group $H_1(S)$ for any compact Riemann surface associated with an arbitrary irreducible plane algebraic curve.

Deconinck and van Hoeji [4] presented codes to perform this algorithm. The "algcurves" package of "Maple" is available to obtain a canonical basis of $H_1(S)$ for an arbitrary irreducible algebraic curve with integral coefficients. We shall explain Tretkoff's method to express the elements of the homology group $H_1(S : \mathbf{Z})$ by using the non-singular quartic curve $F_T(x, y, 1) = 16y^4 + (20x^2 - 12)y^2 + 4x^4 - 12x^2 + 1 = 0$. We shall solve the equation

$$16y^4 + (20x^2 - 12)y^2 + 4x^4 - 12x^2 + 1 = 0, \tag{3.4}$$

in $y$. For a usual value of $x$, this equation in $y$ has 4 distinct solutions. There are 8 exceptional values of $x$ for which some of 4 solutions coincide. The condition for such exceptional values are obtained by eliminate $y$ from the equations

$$16y^4+(20x^2-12)y^2+4x^4-12x^2+1 = 0, \frac{\partial}{\partial y}(16y^4+(20x^2-12)y^2+4x^4-12x^2+1) = 0.$$

The exceptional points are called *branch points* of the curve $F_T(x, y, 1) = 0$. In this case those are solutions of the equation

$$(3x^2 + 5)(3x^2 + 1)(2x^2 + 4x + 1)(2x^2 - 4x + 1) = 0.$$

We denote those branch points by $P_j$ as the following:

$$\begin{aligned}
&P_1 : x = -i\sqrt{5/3} \approx -1.29099i, \quad P_2 : x = -\frac{i}{\sqrt{3}} \approx -0.577350i,\\
&P_3 : x = \frac{-2-\sqrt{2}}{2} \approx -1.70711, \quad P_4 : x = \frac{-2+\sqrt{2}}{2} \approx -0.292893,\\
&P_5 : x = \frac{2-\sqrt{2}}{2} \approx 0.292893, \quad P_6 = \frac{2+\sqrt{2}}{2} \approx 1.70711,\\
&P_7 : x = \frac{i}{\sqrt{3}} \approx 0.577350i, \quad P_8 : x = i\sqrt{5/3} \approx 1.29099i.
\end{aligned}$$

We use the following decomposition

$$\{(x, y) \in \mathbf{C}^2 : F_T(x, y, 1) = 0, x \neq P_j \, (j = 1, 2, 3, 4, 5, 6, 7, 8)\}$$

$$= \cup_{1 \le k \le 8}\{(x,k) : x \in \mathbf{C}, x \neq P_j \ (j = 1,2,3,4,5,6,7,8)\}.$$

We take a base point $x_0 = -2.27279$. If $x_1$ is a complex number and $\Im(x_1) \neq 0$, $x_1 \neq P_j$ $(j = 1, 2, \ldots, 8)$, then the line segment $[x_0, x]$ does not intersect with any of $P_j$. The equation $F_T(x_0, y, 1) = 0$ has the following solutions

$$y_1(-2.27279) \approx -2.26981i,\ \ y_2(-2.27279) \approx -0.744951i,$$

$$y_3(-2.27279) \approx 0.744951i,\ \ y_4(-2.27279) \approx 2.26981i.$$

The values of $y_j(x_0)$ are pure imaginary, and labeled as

$$\Im(y_1(x_0)) < \Im(y_2(x_0)) < \Im(y_3(x_0)) < \Im(y_4(x_0)).$$

The 4 solutions of $F_T(x_1, y, 0) = 0$ are labeled as $y_j(x_1)$ if $y_j(x_1)$ is the analytic continuation of $y_j(x_0)$ along the line segment $[x_0, x_1]$. Concerning the labeling of $y_j(x_1)$ for the points $x_j \in \mathbf{R}$, $x \neq P_3, P_4, P_5, P_6$ are due to Tretkoff's rule. We shall define some closed paths on the curve (3.4) which we use for the computation of the Riemann matrix. Firstly, we define the closed paths $\{a_1, a_2, a_3, b_1, c_6 - a_1, c_7\}$ which start at $x_0 = -2.27279$ and arrive at $x_0$ in the following way:

1. The closed path $a_1$ starts on sheet 1, encircle branch point $P_1$ to arrive at sheet 2, encircle branch point $P_2$ to arrive at sheet 1.

2. The closed $a_2$ starts on sheet 1, encircle branch point $P_4$ to arrive at sheet 4, encircle branch point $P_5$ to arrive at sheet 1.

3. The closed path $a_3$ starts on sheet 3, encircle branch point $P_2$ to arrive at sheet 4, encircle branch point $P_7$ to arrive at sheet 3.

4. The closed path $b_1$ starts on sheet 1, encircle branch point $P_1$ to arrive at sheet 2, encircle point $P_8$ to arrive sheet 1.

5. The closed path $c_6 - a_1$ starts on sheet 1, encircle branch point $P_2$ to arrive at sheet 2, encircle branch point $P_3$ to arrive at sheet 3, encircle branch point $P_7$ to arrive at sheet 4, encircle branch point $P_4$ to arrive at sheet 1.

6. The closed path $c_7$ starts on sheet 1, encircle branch point $P_1$ to arrive at sheet 2, encircle branch point $P_3$ to arrive at sheet 3, encircle branch point $P_8$ to arrive at sheet 4, encircle branch point $P_4$ to arrive at sheet 1.

The closed paths $b_2, b_3$ are defined by $b_2 = b_1 - a_1 - 2(c_6 - a_1) + c_7$, and $b_3 = a_2 + (c_6 - a_1) + a_1 - c_7$. Then the set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ is a canonical basis for the homology group $H_1(S)$. We use another canonical basis $\{\tilde{a_1} = a_1, \tilde{a_2} = a_2, \tilde{a_3} = a_3 - a_2, \tilde{b_1} = b_1, \tilde{b_2} = b_2 + b_3 - a_2, \tilde{b_3} = b_3\}$. Then the cycles $\tilde{b_2}, b_3$ satisfy $\tilde{b_2} = b_1 - (c_6 - c_1), \quad b_3 = a_2 + (c_6 - a_1) + a_1 - c_7$.

We shall compute the Riemann matrix $(r_{ij})_{i,j=1}^4$ with respect to the canonical basis

$$\{\tilde{a_1}, \tilde{a_2}, \tilde{a_3}, \tilde{b_1}, \tilde{b_2}, \tilde{b_3}\}.$$

The original $\{a_1, \ldots, b_3\}$ is the basis of $H_1(S)$ which the "algcurves" package provides. But we replaced the original basis by another one to get a Riemann matrix expressed in a good form. We compute the $A$-matrix $(a_{ij})$, and the $B$-matrix $(b_{ij})$ defined by

$$a_{ij} = \oint_{\tilde{a_j}} \omega_i, \quad b_{ij} = \oint_{\tilde{b_j}} \omega_i.$$

These matrices $A = (a_{ij}), B = (b_{ij})$ are invertible. The Riemann matrix $R = (r_{ij})$ of the Riemann surface $S$ is defined as $A^{-1}B$ which is a complex symmetric matrix and $\Im(R)$ is a positive definite (real) symmetric matrix for a general Riemann surface $S$. The Riemann matrix is an invariant to characterize the metric structure of the period lattice of a Riemann surface $S$. By using the $g \times g$ matrix $R = (r_{ij})$, we can embed the homology group $H_1(S)$ in the vector space $\mathbf{C}^g$. Let $\{\omega_1, \ldots, \omega_g\}$ be a basis of the cohomology group $H^1(S)$.

For any $\gamma \in H_1(S)$, consider a vector

$$(\gamma(1), \cdots, \gamma(g)) \in \mathbf{C}^g, \text{where}\, \gamma(i) = \oint_\gamma \omega_i$$

( $i = 1, \ldots, g$). In this way, $H_1(S)$ viewed as a lattice $\Gamma$ in $\mathbf{C}^g$ isomorphic to $\mathbf{Z}^{2g}$.

Let $R = \{e_{g+1}, \ldots, e_{2g}\}$. Let $e_1 = \{1, 0, \ldots, 0\}^T, \ldots, e_g = \{0, \ldots, 0, 1\}^T$ be the standard basis of $\mathbf{C}^g$. Consider the standard inner product on $\mathbf{C}^g$. The $2g$ vectors $\{e_1, \ldots, e_g, e_{g+1}, \ldots, e_{2g}\}$ are generators of the lattice $\Gamma$. We wish to characterize the metric structure of $\Gamma$ for a Riemann surface $S$ for $F_T = 0$ of a matrix $T$.

**Proposition 3.1.** (Chien, N. [2]) Let $T = \begin{pmatrix} 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then the genus of the curve $F_T = 0$ is 3. For a suitably chosen cycles $a_i, b_j$, we can decompose $\Gamma$ as the orthogonal direct sum of two lattices $\Gamma_1$ spanned by $e_1, e_2, e_6$ and $\Gamma_2$ spanned by $e_3, e_4, e_5$: $\Gamma = \Gamma_1 \oplus \Gamma_2$.

Numerical approximations of these vectors are given by

$$e_1 = (1,0,0)^T, e_2 = (0,1,0)^T, e_3 = (0,0,1)^T$$

$$e_4 = (1.69486i, 0.847432i, 0.152568)^T, e_5 = (0.847432i, 1.54696i, 0.576284)^T,$$
$$e_6 = (0.152568, 0.576284, 0.576284i)^T.$$

The vector $e_6$ is orthogonal to $e_3, e_4, e_5$. The vectors $e_4, e_5$ are orthogonal to $e_1, e_2$.

**Proposition 3.2.**(Chien, N., [2]) Let

$$\tilde{T} = \begin{pmatrix} 0 & 2 & 2a & 2k \\ 0 & 0 & 2 & 2a \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with real $a > \sqrt{2}$ and $k = a^2 - 1$. Then the genus of the curve $F_T$ is 2. For a suitably chosen $\tilde{a_i}, \tilde{b_j}$, the$A$-matrix is a real matrix and the $B$-matrix is a pure imaginary matrix and hence $R$ is a pure imaginary matrix. The lattice $\Gamma$ is the orthogonal direct sum of $\Gamma_1$ spanned by $e_1, e_2$ and $\Gamma_2$ spanned by $e_3, e_4$.

For the proof of Proposition 3.2, we choose the canonical basis $\{\tilde{a_1}, \tilde{a_2}\tilde{b_1}, \tilde{b_2}\}$ as the following. Let $\{a_1, a_2, b_1, b_2\}$ be the canonical basis for the homology group $H_1(\Gamma)$ produced by the algorithm in [4] and the algcurves implementation. We construct a new basis depending on the canonical basis as follows:

1. The cycle $\tilde{a_1} = a_1$ starts on sheet 1, encircles branch point $r_1 = (a^2 - \sqrt{a^4 + 8a + 8})/(4(a+1))$ to arrive at sheet 2, encircle branch point $r_2 = (a^2 - \sqrt{a^4 - 8a + 8})/(4(a-1))$ to arrive at sheet 1.

2. The cycle $\tilde{a_2} = -a_2$ starts on sheet 1, encircle branch point $r_3 = (a^2 + \sqrt{a^4 + 8a + 8})/(4(a + 1))$ to arrive at sheet 2, encircle branch point $r_4 = (a^2 - 2a + 2)/(2(a - 1))$ to arrive at sheet 1.

3. The cycle $\tilde{b_2} = -b_2 - a_1$ starts on sheet 1, encircle branch point $r_4 = (a^2 - 2a + 2)/(2(a - 1))$ to arrive at sheet 2, encircle branch point $r_5 = (a^2 + 2a + 2)/(2(a + 1))$ to arrive at sheet 1.

4. The cycle $\tilde{b_3} = \tilde{b_1} - \tilde{b_2} = b_1 + b_2 + a_2$ starts on sheet 1, encircle branch point $r_2 = (a^2 - \sqrt{a^4 - 8a + 8})/(4(a - 1))$ to arrive at sheet 2, encircle branch point $r_3 = (a^2 + \sqrt{a^4 + 8a + 8})/(4(a + 1))$ to arrive at sheet 1.

Consider the set $\{\tilde{a_1}, \tilde{a_2}, \tilde{b_1}, \tilde{b_2}\}$ in the group $H_1(\Gamma)$ given by

$$\tilde{a_1} = a_1, \quad \tilde{a_2} = -a_2, \quad \tilde{b_1} = b_1 - a_1 + a_2, \quad \tilde{b_2} = -b_2 - a_1.$$

The new basis $\{\tilde{a_1}, \tilde{a_2}, \tilde{b_1}, \tilde{b_2}\}$ of $H_1(\Gamma)$ is suitable for our aim.

In the case $a = 2$, the matrices $A, B$ are approximately given by

$$A \approx \begin{pmatrix} -0.195915 & 0.410645 \\ -0.0199325 & 0.382071 \end{pmatrix}, \quad B \approx \begin{pmatrix} 0.170162i & 0.655061i \\ 0.553593i & 0.865335i \end{pmatrix}.$$

# 4. Development of the study of Riemann matrices

We shall briefly mention the history of the study of Riemann surfaces. Bernhard Riemann (1826-1866) built the foundation of Riemann surfaces (complex analytic 1-dimensional manifolds). Our main interests consists in compact Riemann surfaces. His papers [9, 10, 11] are classical literatures of this subjects. Torelli [12] showed that the Riemann matrices are complete invariants of compact Riemann surfaces. Some Japanese mathematicians are studying Riemann matrices (cf. [6], [14]). Recently K. Konno published a nice introduction to the theory of Riemann surfaces and algebraic curves [8]. To study Riemann surfaces, some computer softwares help us to treat this matrix. The above results would be just a start point of the study of Riemann surfaces related to the numerical ranges.

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